Research Article

A Proof Showing That The Collatz Conjecture Holds For All Positive Integers

Abstract: A proposed proof of the Collatz Conjecture is presented showing the conjecture holds true for all positive integers. The proof will create a modified version of the conjecture, specifying that rather than performing successive divisions of even integers, a single modified Collatz equation omitting the division step may be applied to see if the graph thus produced will ultimately produce numbers that are some exponent of 2.

Keywords: Numerical Theory, Collatz Conjecture

Introduction:

The Collatz conjecture postulates that for any integer, the following rule set applied successively to the solution of the previous application, will eventually yield an answer of one:

-If a number is even, divide by two.

-If the number is odd, triple it and add one

The main problem with proving the Collatz Conjecture is the use of two rules, and the decision of which one to use being dependent upon whether the result of the last rule execution was a positive or negative integer, something which has, until this point, been seen as a random outcome. This prevents the generation of a smooth graph, which could be examined using limit rules.

For this proof, we will begin by looking at what the Collatz Conjecture is really doing, and create a single analogous function to perform the same task, comprised of one rule which may be run continuously regardless of whether the result of the last run was positive or negative. Thus we may then take the equation to its limit, and see if it will do what the Collatz Conjecture does, for all integers up through infinity.

We will then use the modified equation to examine how the Collatz Conjecture is generating the root odd numbers necessitating the application of the odd rule, and show how this analysis will reveal exactly why that root off number always goes to zero for all integers.

Analysis:

The first step is to recognize that the repeated divisions by two of every even result, in what will be referred to as the Old Collatz equation, are not necessary, an indeed impede a clear picture of what the process is doing. In addition, one need not waste time examining even integers, since they will just immediately be divided down to an odd integer in Old Collatz, at which point the odd integer rule will begin being applied.

Thus, you can begin with any odd integer and then continue to perform the step you perform on odd integers, multiplying it by three and increasing the extra amount added commensurately at each step, to reflect that you never divided by two before adding the one.

What the Old Collatz conjecture's division by two of even integers with an objective of ultimately attaining an answer of one essentially does is test if this New Collatz rule, applied to any odd integer will eventually yield a result represented by 2^x, where x is any integer, which when divided by 2 repeatedly will eventually yield an answer of one.

In traditional Collatz, you perform the divisions by two as you go along applying the odd integer rule, until you get the answer "one." In the new Collatz Equation, we will merely perform the triple multiplication and addition step repeatedly in the correct relative proportions, and see if that equation, taken to the limit of Infinity will eventually coincide with points somewhere on a graph of $y=2^x$. If it does, it should then be obvious that repeatedly divided by two, it will eventually yield one.

We will also take a deeper look at the mechanism underlying Collatz, and show why this occurs, which should answer once and for all that the Collatz Conjecture applies to all positive integers.

The New Collatz Equation can be represented by:

$$y=3(x) + 2^m$$

where $x=(2^m)(z)$, where z is an odd number. Merely divide x by 2 repeatedly until you get an odd number, to find z. The number of divisions necessary will serve as m. Obviously m can equal zero if necessary, such as at the beginning of processing an odd integer.

It is derived by assuming we began with an odd integer represented by (n+1) (n being an even number), multiplied it by 3 and added one, and then neglected to divide by two. That leaves us with:

$$3(n+1)+1$$
$$\downarrow$$
$$3n+4$$

If we perform the step a second time we must remember that because we never divided by 2, the number we add must be multiplied by two, ergo:

$$3[(3n+4)]+2$$

Notice, if given any even number, every division by two neglected is contained within the 2^m outlined above. Since in New Collatz, the ultimate product of the one added in Old Collatz is multiplied by two for each division neglected, the number added in each New Collatz equivalent (3x+1) step may be represented simply by 2^m.

Thus the new Collatz Equation, which can be applied repeatedly regardless of whether a number is even or odd, becomes:

$$y=3x+2^m$$

The second point to note is that each odd step of the Old Collatz, and each step of the New Collatz will both add at least one integer to exponent m. This can be seen in Old Collatz, where an odd integer is multiplied by three, yielding an odd integer, and then a one is added, making it even and divisible by two, thus adding one to the exponent of two. The same operation occurs in New Collatz, down beneath the two's by which you have neglected to divide the old products.

If you examine the effect on z, the base odd number, in the New Collatz, after removing the twos which we have failed to divide by, you will see the following, which will be the base odd number we are really dealing with, and how it is handled in each step of the New Collatz:

$$y_1=3x + 2^m$$

$$\downarrow$$

$$y_1=3(2^m)(z)) +2^m$$

$$\downarrow$$

$$y_1=3z2^m +2^m$$

removing the twos we have failed to divide by (2^m) yields:

$$3z+1$$

Obviously that is the old Collatz factor, and has the same effect of producing an even number, allowing the divisor two to be pulled up into 2^m. It also shows that beneath the undivided twos, New Collatz is performing the same essential function of old Collatz.

If one can accept each step of the cycle would increase m by at least one, it should be noted New Collatz, performed an infinite number of times, would look as follows as it reached infinity:

$$y= 3(2^{\infty} * z) +2^{\infty}$$

Thus taken to infinity, the odd integer z should diminish in relevance and the New Collatz will coincide with:

$$y= 3(2^x) +2^x$$
$$\downarrow$$
$$y=4(2^x)$$

Obviously the product of any number on that graph, divided by two an infinite number of times, will eventually yield one.

The proof is still incomplete, as one might argue that z could approach infinity as well, competing with 2^m, even though as an odd integer it should not. Regardless of the infinity issue, this is not the case, and in fact, Z inevitably goes to one. We will finish the proof by explaining why that is, and what Collatz is ultimately doing as it approaches $4(2^x)$, and what mechanism lies beneath the organization of the series of base odd numbers in every set, and why that organization and the mathematical processing of it drives z to 1.

In short, from the perspective of the Collatz Conjecture, these root odd numbers are actually being manipulated based upon an alternate representation of them as additive products of 1 and a series of varied and increasing powers of 2. Three examples of such a modified representation follow:

$$25=1+2^3+2^4$$

or

$$19=1+2^2+2^4$$

or

$$43=1+2^1+2^3+2^5$$

If you attempt to perform a New Collatz step on such a numerical representation, it is still difficult to see what is going on, because the introduction of the number three, with a somewhat random and arbitrary nature as a unique number in its own right, obscures the mechanism. So to clarify it, in the next example, we will alternately represent the 3 as (1+2) in places where it will be most illustrative below, as we perform a representation of such a New Collatz sequence for three steps on an odd number represented as $(1+2^a)$.

The important part will be what happens to a, although it is worth noting on the initial odd number 1 in the set, the New Collatz merely turns it into a 2^2 in its first step after running, by multiplying it by three and adding one. This occurs as well in subsequent runs, and is key to why z eventually heads to one. In each cycle that x^2 allows the division of the entire representation by at least x^2,(assuming no consolidation of exponents, which can allow division by an even larger exponent). How this reduces the overall value of all the exponents will become apparent. The actual numerical value of this representation of z, here expressed in 1 plus a series of exponents of two, is slowly being lowered, only rising temporarily intermittently, as consolidation of similar exponents temporarily raises the overall value of z by creating a single temporarily elevated power of two component, before the process of exponent reduction continues. Now, the example:

$$3x+2^m$$

$$\downarrow$$

$$3(1+2^a) + 2^0$$

$$3+1+3(2^a)$$

$$4+(1+2)(2^a)$$

$$4+(2^a+2^{a+1})$$

$$2^2+(2^a+2^{a+1})$$

$$2^2+2^a+2^{a+1}$$

$$2^2\,(1+2^{a-2}+2^{a-1})$$

$$\downarrow$$

$$3[2^2(1+2^{a-2}+2^{a-1})]\ +2^2$$

$$2^2[3[(1+2^{a-2}+2^{a-1})]+1\ ]$$

$$2^2[3+(1+2)(2^{a-2}+2^{a-1})+1]$$

$$2^2[2^2+(2^{a-2}+2^{a-1}+2^{a-1}+2^a)]$$

$$2^4[1+(2^{a-4}+[2^{a-3}+2^{a-3}]+2^{a-2})]$$

$$2^4[1+(2^{a-4}+[2^{a-2}+2^{a-2}])]$$

$$2^4[1+(2^{a-4}+2^{a-1})]$$

$$\downarrow$$

$$3[2^4[1+(2^{a-4}+2^{a-1})]]+2^4$$

$$2^4[3+[(1+2)2^{a-4}+(1+2)2^{a-1}]]+1]$$

$$2^4[3+(2^{a-4}+2^{a-3})+(2^{a-1}+2^{a})]]+1]$$

$$2^4[2^2+2^{a-4}+2^{a-3}+2^{a-1}+2^{a}]$$

$$2^6[1+2^{a-6}+2^{a-5}+2^{a-3}+2^{a-2}]$$

So the pattern is, a base odd number ($1+2^a$) with exponent a will, with a first cycle, see the next base odd number's exponential component become $1+2^{a-2}+2^{a-1}$, where the primary exponent a will be reduced to a-1, with an additional fragment 2^{a-2} added. On the next cycle, because of the proximity of the exponents, there will be an exponent merge, as 2^{a-2} will become 2^{a-4} and 2^{a-3}, as 2^{a-1} becomes 2^{a-2} and 2^{a-3}, which will merge powers of 2 and produce 2^{a-1} and 2^{a-4}. These powers of 2 will have enough distance between them to do an unmerged reduction, as 2^{a-4} will become 2^{a-6} plus 2^{a-5}, and 2^{a-1} will become 2^{a-3} and 2^{a-2}.

Obviously this pattern becomes less easily predicted when there are multiple exponents spaced closely enough to interact before each is extracted and removed. You will notice in the example at the end of this paper, only numbers represented by two expoenents at the beginning, such as 7 ($1+2^1+2^2$) and 11 ($1+2^3+2^2$) rise, while numbers represented with a single exponent such as 17 ($1+2^4$) or 5 ($1+2^2$), or represented by expoenents far enough away they were removed before merging such as 13 ($1+2^1+2^3$) would reliably be reduced in size by the next cycle. Exponents will only increase if they merge, and after merging, they will all inevitably reduce to 0.

m is increasing steadily, drawing exponents of two out of z's modified representation and depositing them in m. Any exponent of two within the brackets will, over repetitive executions of the cycles, reduce and expand below itself into other lower powers, with lower exponents periodically merging and raising. But overall, the exponents in the entire set are gradually decreasing to zero, at which point these zero exponents will merge with the 1 into a 2, allowing one final exponent of one to be pulled out to exponent m. As this occurs, each power-of-two component of z is reduced to a lower power of two, and z is reduced in size. Eventually you will have only one exponent of zero left with a one, which will combine to two, and a final exponent can be withdrawn and deposited in m, leaving 2^0 within the brackets, multiplied by 2^m.

For illustrative purposes, if a had equaled 4, and thus x had equaled 17 at the start, this process would appear as follows:

$$3x+2^m$$

$$\downarrow$$

$$3(2^0+2^4) + 2^0$$

$$3+1+3(2^4)$$

$$3+1+(1+2)(2^4)$$

$$4+(2^4+2^5)$$

$$2^2+(2^4+2^5)$$

$$2^2+2^4+2^5$$

$$2^2+(1+2^2+2^3)$$

$$\downarrow$$

$$3[2^2(1+2^3+2^2)] +2^2$$

$$2^2[3[(1+2^2+2^3)]+1 \ ]$$

$$2^2[3+(1+2)(2^2+2^3)+1]$$

$$2^2[2^2+(2^{a-2}+2^{a-1}+2^{a-1}+2^a)]$$

$$2^4[1+(2^0+[2^1+2^1]+2^2)]$$

$$2^4[1+1+2^2+2^2)]$$

$$2^4[2^1+2^3)]$$

$$2^5[1+2^2)]$$

$$\downarrow$$

$$3[2^5[1+2^2)]] +2^5$$

$$2^5[3+(1+2)2^2]] +1]$$

$$2^5[4+(2^2+2^3)]$$

$$2^5[2^2+2^2+2^3]$$

$$2^5[2^3+2^3]$$

$$2^5[2^4]$$

$$2^9[1]$$

You can easily relate this process above directly to the application of the Collatz Conjecture to the number 17, by using either Old Collatz and focusing on the odd numbers and the total number of divisions done up until they are attained, which will match at each step, or using New Collatz and relating the exponent m and the base odd number z to its correlate at each stage.

Where you begin with multiple different powers of two as components of a number, perhaps many in the case of a very large number undergoing a large number of runs, the process may become more messy as subsequent steps merge various lower exponents, raising them in a single higher exponent that temporarily elevates z's size, only to drop them again even lower and shrink z further. It is this complex aspect of the merging of exponents to a single raised exponent, which is responsible for the seemingly random nature of the sequence of base odd numbers Old Collatz would throw out successively, as well as the periodic rise of a base odd number, before it would resume dropping. This was the order behind the seeming chaos.

Eventually after the complex merging and reducing in number of nearly randomly arrayed and interacting exponents occurs, and the smoke clears with one exponent left, the process will continue and the exponents will all gradually be merged and reduced to zero, until all that remains within the parenthesis is a 1 multiplied by the 2^m, which in the next run of the cycle will become $(3)2^m+2^m$, or $4(2^m)$. It has then merged with $y=4(2^x)$, and can then be divided down to one smoothly.

In closing, I will offer below one example below of Old Collatz alongside New Collatz, displayed alongside $y=2^x$ as a demonstration of these curves coinciding at smaller scales, as well as an example of how the essence of Old Collatz is contained within New Collatz, and it is ultimately following the same numerical path, merely more directly in one direction, before turning and following it the exact same path as Old Collatz in the other direction:

Number Seven

Old Collatz		New Collatz		$y+2^x$	
7		7		2	
22				4	
11	(1 total division)	22	($11x2^1$)	8	
34				16	
17	(2 divisions)	68	($17x2^2$)		32
52				64	
26		208	($13x2^4$)		128
13	(4 divisions)			256	
40		640	($5x2^7$)	512	
20				1024	
10		2048	(2^{11}, thus 11 divisions to 1)	2048	
5	(7 divisions)			4096	
16		8192 (2^{13})	$y=4(2^x)$	8192	
8					
4					
2					
1	(11 total divisions)				